Guaranteed Reachability for Systems with Impaired Dynamics

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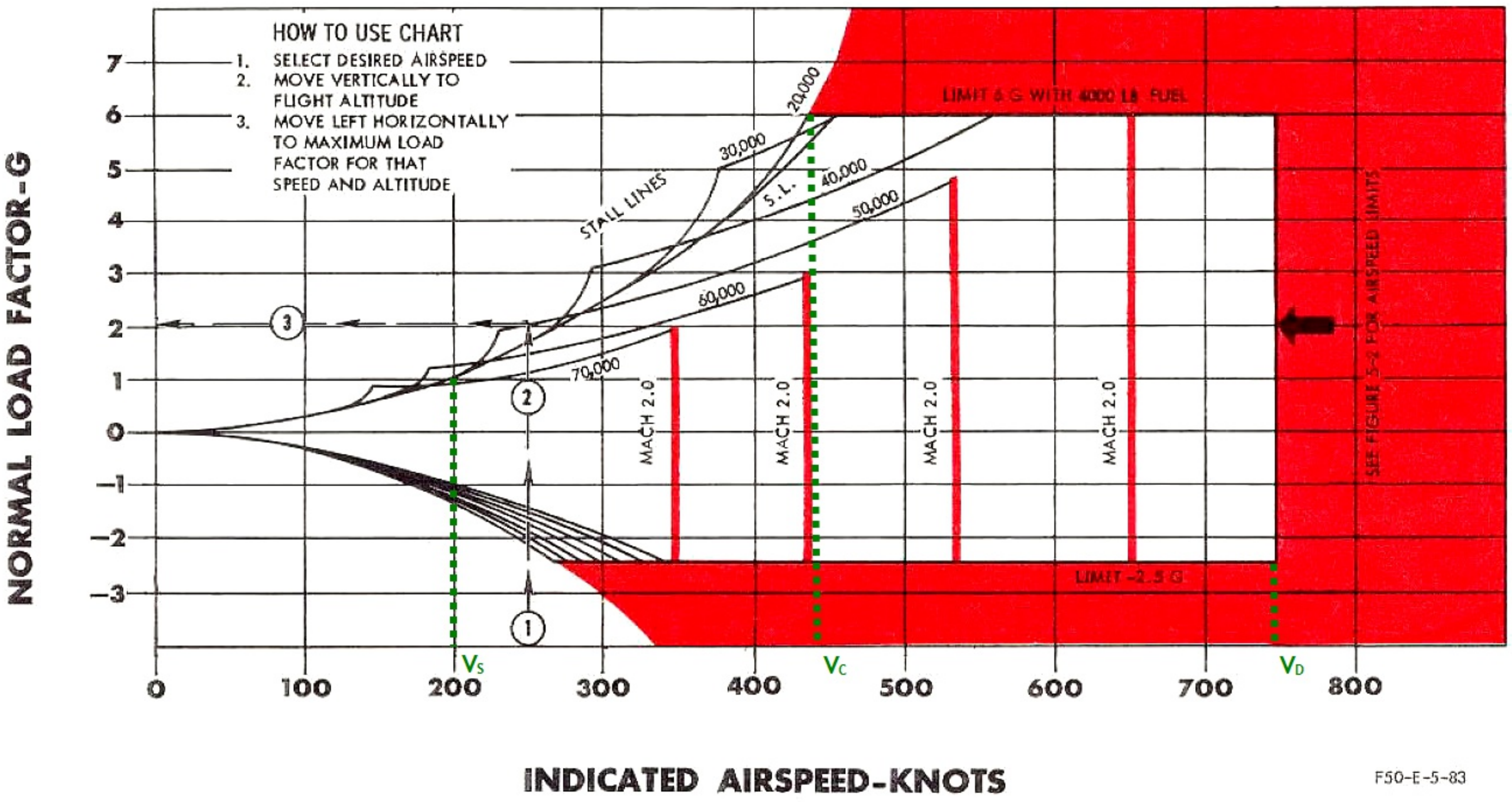
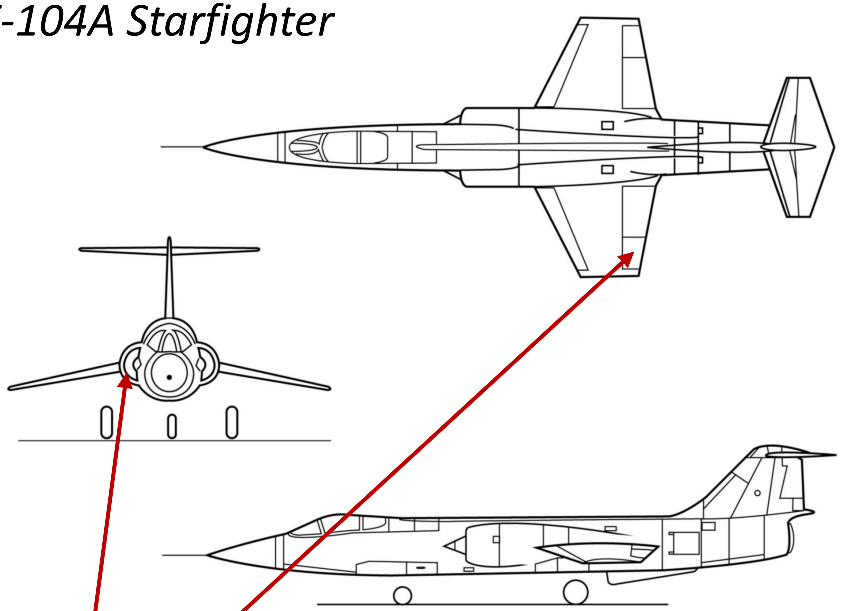
Motivation

Flight envelope



F-104A Starfighter



What happens if my ailerons degrade?

What can I still do when I'm low on fuel?

Motivation

- Producing reachable sets is **hard**.
- Despite failures, we still want to know our **system's *guaranteed* capabilities**:
  - A priori computation of reachable sets is impossible when facing **dynamic failure modes**.
- Current approaches to reachable set computation focus mainly on *outer approximations*:
  - Outer approximate reachable sets are more optimistic and are **not guaranteed to yield viable results**.
  - In this setting, we are interested in *inner approximations*.
- Can we **reuse our prior knowledge** in off-nominal conditions?

Motivation

"What can my system do?"

- Guaranteed capabilities (inner approximation).
- "If nothing else, we can at least do this."
- Useful for safety critical control, such as when experiencing partial failure or off-nominal operating conditions.

UA 328 after right-engine failure (AP)

"What could my system do?"

- Potential capabilities (outer approximation).
- "In the worst case, this could happen."
- Useful for collision avoidance and safety envelope design.

Approach

- We use a **conservative analytical bound** on the change in dynamics of the *off-nominal* system with respect to the dynamics of the *nominal* system.
- We focus on the case of **diminishing control authority**, which requires an upper bound on the **distance between the nominal and off-nominal set of admissible control inputs**.
- We leverage **knowledge of the nominal reachable set**, **reachable set convexity** and a **bound on the minimum trajectory deviation** between trajectories of the nominal and off-nominal reachable set.
- Our approach **shrinks** the known nominal reachable set by a computed distance, yielding an **inner approximation of the impaired reachable set**, making it applicable for **online use**.

Rationale

- When the **nominal reachable set is available**, can we *reuse* it to find an inner approximation of the off-nominal reachable set?
  - Reachable set computation from scratch is expensive and is *not* suitable for **spur-of-the-moment decision making**.
- Changes in the dynamics can be overapproximated, and the **minimum deviation between two trajectories** of the nominal and off-nominal system can be upper bounded using *integral inequalities*.
- *If* both **reachable sets are guaranteed to be convex**, we can *shrink* the nominal reachable set by the upper bound on the trajectory deviation and obtain a guaranteed reachable set of the off-nominal system.

Preliminaries

- Consider a dynamical system with *n* states and *m* control inputs, with an initial time t_0, and a compact admissible set of controls $\mathscr{U} \subset \mathbb{R}^m$:

$$\dot{x}(t) = f(t, x(t), u(t))$$

with $f : [t_0, \infty) \times \mathbb{R}^n \times \mathscr{U} \to \mathbb{R}^n$.

- We consider here the *forward reachable set* (FRS), which is defined by the following components:
  - A set of initial states $\mathscr{X}_0$ at time t_0;
  - A time $t_1 > t_0$;
  - The set of admissible control inputs: $\mathbb{U} = \{\phi : \mathbb{R} \to \mathscr{U}\}$;
  - A set of trajectories of the form $\varphi(t | t_0, x_0, \phi) : [t_0, \infty) \to \mathbb{R}^n$.

Preliminaries – Forward Reachable Set

- We will represent the dynamics in terms of set-valued *multifunctions* of the form $F : [t_0, \infty) \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n = f(\cdot, \cdot, \mathscr{U})$.
- The FRS is defined as $\mathbb{X}_t^{\rightarrow} = \mathbb{X}_t^{\rightarrow}(F, \mathscr{X}_0) := \{\varphi(t|t_0, x_0, \phi) : x_0 \in \mathscr{X}_0, \phi \in \mathbb{U}\}$.

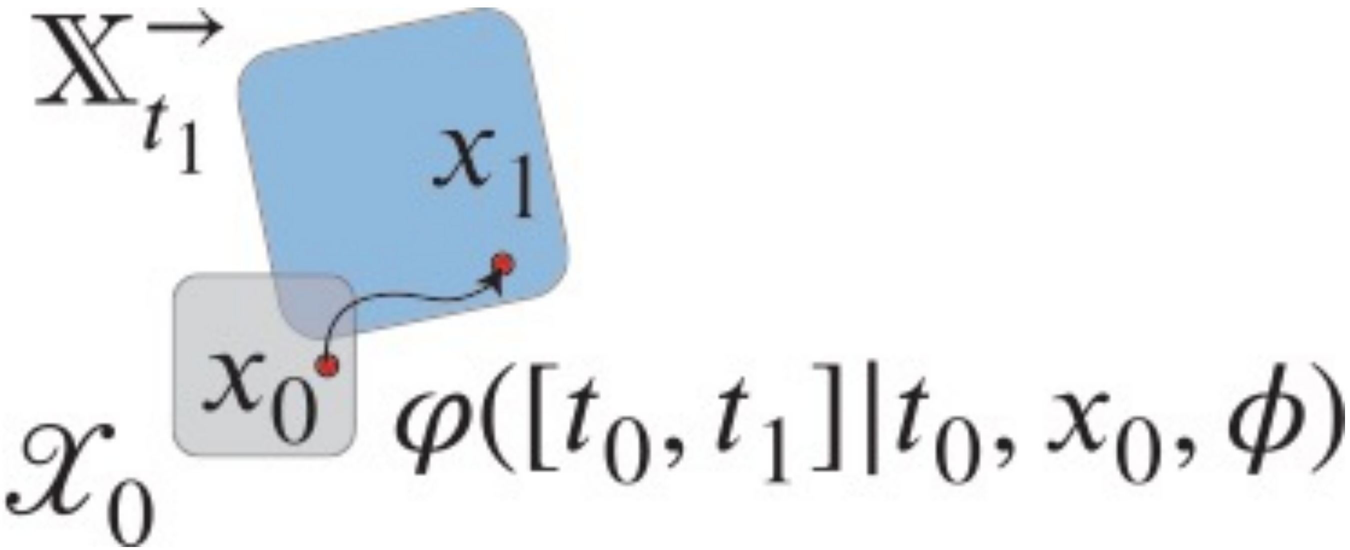


Figure 1: *Illustration of a forward reachable set*

Preliminaries – Diminished Control Authority

- We denote the *impaired* or *off-nominal* counterparts of the nominal system's properties by an overbar.
- In case of diminished control authorit
  - The dynamics remain unchanged;
  - The set of admissible control inputs shri
    $\bar{\mathscr{U}} \subset \mathscr{U}$
  - The off-nominal reachable sets are

    subsets of the nominal reachable sets.

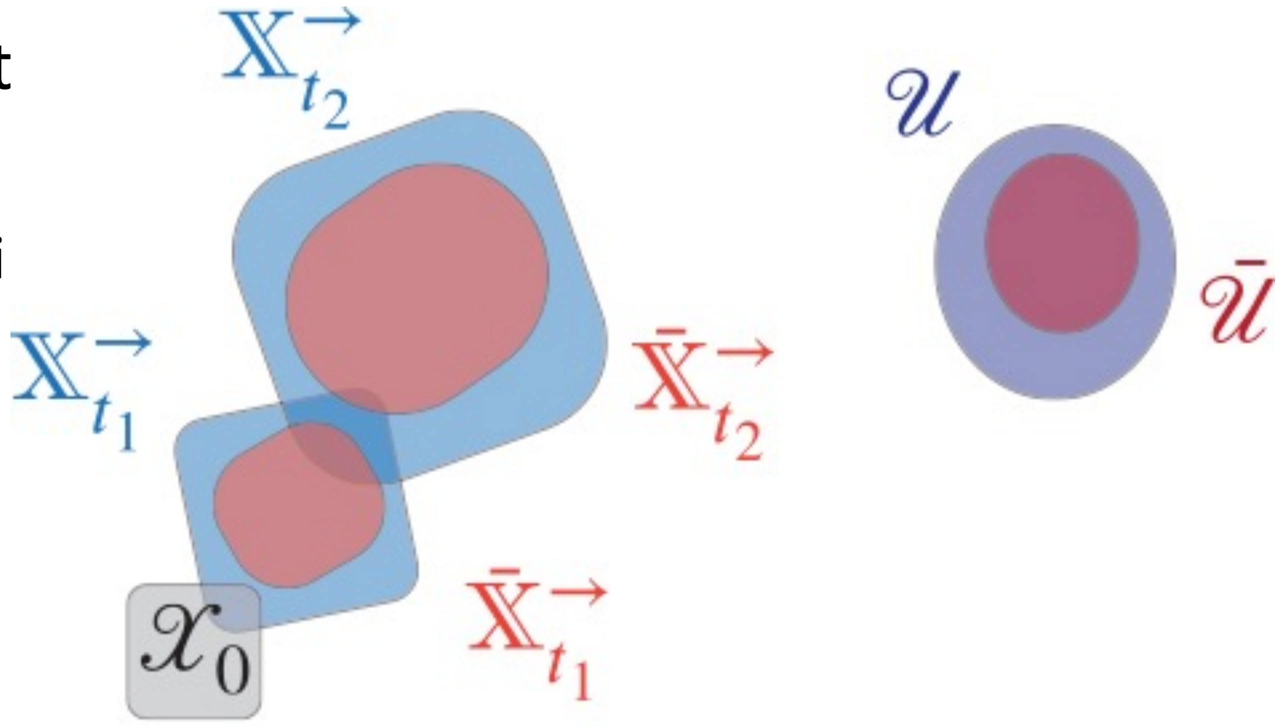


Figure 2: *Illustration of the effect of diminished control authority on the FRS*

Sufficient Conditions for Convexity of FRS

- An *R*-convex set is a compact set that can be constructed as the intersection of balls of radius *R* (this intersection need not be finite or countable).
- A sufficient condition requires that $\mathscr{X}_0$ is R_0-convex, and that F is *R*-convex, as well as some technical growth conditions.
- In general, these conditions hold for affine-in-control systems.

Trajectory Deviation Growth Bound

- We wish to find an **upper bound** $\eta(t)$ on the minimum deviation between a nominal and off-nominal trajectory originating from the same initial state.
- **State-agnostic** trajectory growth bound:

$$\tilde{f}(t) := f(t, x(t), u(t)) - f(t, \bar{x}(t), \bar{u}(t))$$

$$\|\tilde{f}(t)\| \leq \tilde{a}(t)\tilde{w}(\|\tilde{x}(t)\|, \|\tilde{u}(t)\|) + \tilde{b}(t)$$

- By application of a Bihari inequality, we can find:

$$\|\tilde{x}(t)\| \leq \eta(t)$$

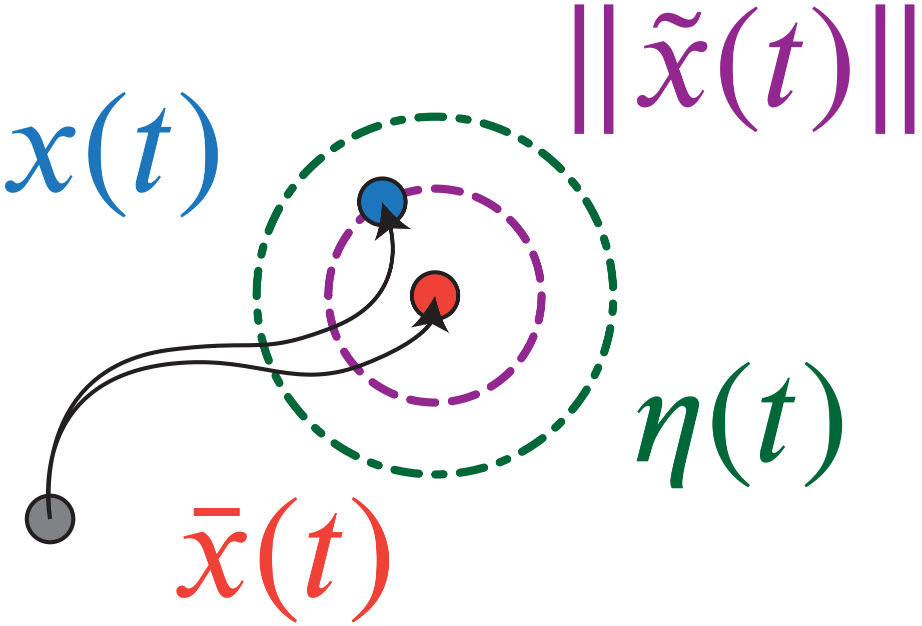


Figure 3: *Illustration of trajectory deviation bound*

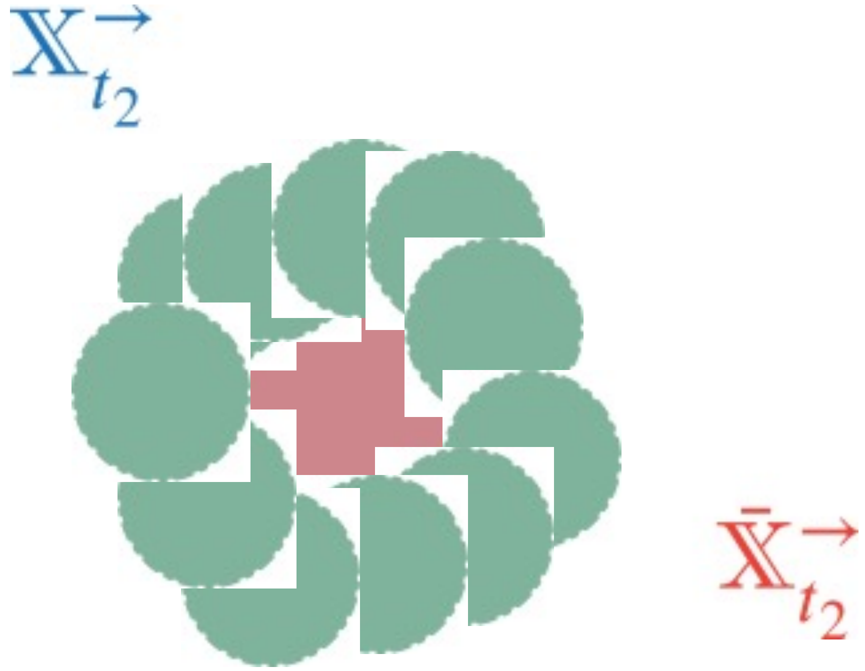
Inner Approximation of the Off-nominal FRS

- We have conditions for which the **FRS is convex**.
- We have an **upper bound on the minimum trajectory deviation** between the nominal and off-nominal FRS.
- Our theory proves that it suffices to **shrink** the

nominal FRS by $\eta(t)$ to obtain an

inner approximation of the off-nominal FRS:

$$\mathbb{X}_t^{\rightarrow} \setminus \left(\bigcup_{x \in \partial \mathbb{X}_t^{\rightarrow}} \mathcal{B}(x, \eta(t)) \right) \subseteq \bar{\mathbb{X}}_t^{\rightarrow}$$



Application – Wing Rock

- A phenomenon of aircraft flying at high-angle of attack, e.g., fighter jets.
- Flow asymmetries cause the aircraft to ‘rock;’ this can lead to **loss of control**.
- Our setting: aileron deflection has become less effective at high angle of attack. This results in diminished control authority.

$$f(x,u) = \dot{x} = \begin{bmatrix} \dot{\phi} \\ \dot{p} \end{bmatrix}$$

$$= \begin{bmatrix} p \\ \theta_1 \phi + \theta_2 p + (\theta_3 |\phi| + \theta_4 |p|)p + \theta_5 \phi^3 \end{bmatrix} + \begin{bmatrix} 0 \\ \theta_6 \end{bmatrix} u$$

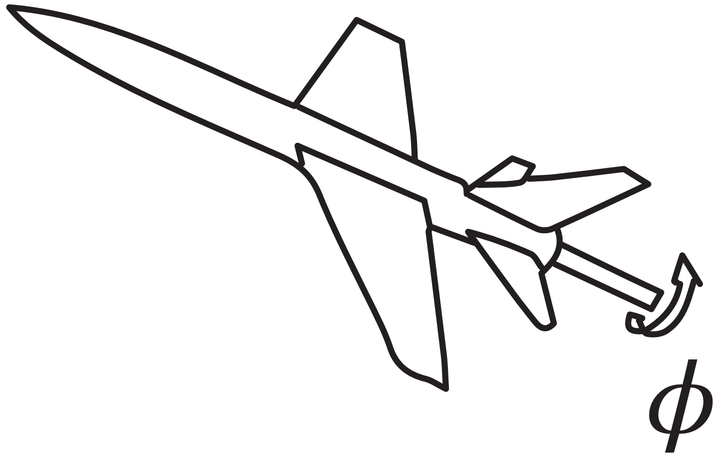


Figure 4: *Illustration of wing rock*

Application – Wing Rock

- We consider +/- 10 degrees aileron deflection nominally, but off-nominally:
  - 15% decrease in stick-forward aileron authority;
  - 5% decrease in stick-backward authority.
- We have the following trajectory deviation growth bound:

$$\|\bar{f}(\bar{x},\bar{u})\| \leq \|\bar{x}\| \left[1 + |\theta_1| + |\theta_2| + (|\theta_3| + |\theta_4|)(2M + \|\bar{x}\|)\right]$$
$$+ |\theta_5|\|\bar{x}\|^3 + |\theta_6|\|\bar{u}\|$$

$$M := \max_{y \in \mathbb{X}_t^{\rightarrow}(F,\mathcal{X}_0)} \|y\|$$

Application – Wing Rock

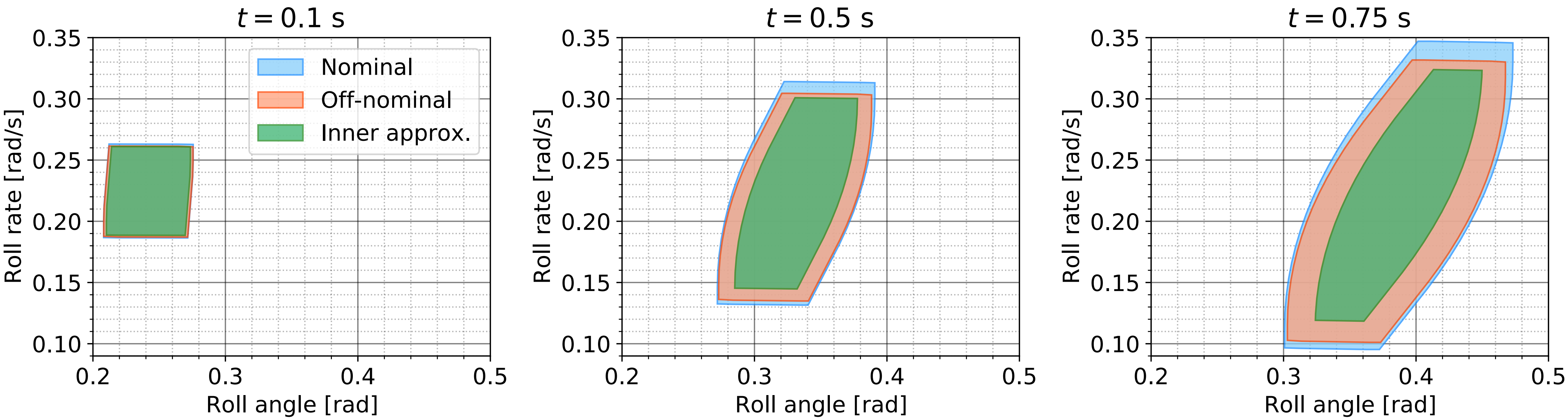


Figure 5: *Application of reachable set inner approximation to wing rock*

On the Horizon

- The theory is easily extensible to (time-varying) changes in dynamics using the Bihari inequality.
    - This allows for applications to drones with defective rotors to plan safe landing trajectories, or road vehicles to come to a safe stop when experiencing adverse road conditions.
- It is possible to obtain outer approximations of the reachable set by expanding (instead of shrinking), without the need for convexity.
    - This opens up avenues for just-in-time collision avoidance with uncertain moving targets, or preventing vehicles from entering an unsafe state.
- We are working on relaxing system constraints for inner approximation using model order reduction and other approximation techniques.

Closing Remarks

- Reachability analysis brings many new possibilities to life when it comes to safe control and autonomy, especially when it can be performed online.
- This research was done in collaboration with Dr. Melkior Ornik with the LEADCAT group: https://mornik.web.illinois.edu/
- A simple demo can be found on GitHub: https://github.com/helkebir/Reachable-Set-Inner-Approximation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